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LETTER TO THE EDITOR

Percolation of hypersurfaces and finite-size scaling

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Abstract. We consider the percolation of a surface without holes in a system of randomly occupied plaquettes. We show that duality can relate not only the critical points but also the correlation length exponents of this problem to the usual random bond percolation. The generalisation to hypersurfaces in higher dimensions is discussed.

In random bond percolation one usually considers the threshold p_c^B as the smallest probability at which a spanning path appears in a box of arbitrary size. At this critical point, the backbone of the infinite cluster appears, which can be considered as a union of spanning self-avoiding walks. One can also ask the question whether in three dimensions a singly connected (no holes) surface appears spanning through the infinite system. This surface percolation is of interest because of its connection to lattice gauge theory (Aizenman *et al* 1983) and it is closely related to the problem of fracture threshold (Chelidze 1980). However, it is different from the plaquette problem of Wilke *et al* (1985), which seems to be equivalent to a site percolation model where connections up to the fourth neighbours are allowed.

Aizenman *et al* (1983) gave arguments that, for $d=3$ in a system of randomly occupied plaquettes, at a given value of the concentration p^S of plaquettes, there is a transition in the asymptotic behaviour of the probability of finding a large area on surfaces without holes inside. They also showed that the critical point p_c^S of this problem is related to the critical point p_c^B of the bond percolation threshold by duality and $p_c^S = 1 - p_c^B$. In this letter we first discuss the critical behaviour of this surface percolation transition in a cubic lattice and then its generalisations to higher-dimensional hypercubic cases.

The difficulty in this surface percolation arises from the fact that the clusters cannot be defined unambiguously. A surface with holes inside (essentially the problem of Wilke *et al* 1985) cannot be considered as a single cluster, since it can be infinitely large far before the threshold of singly connected surfaces we are interested in. Therefore, the holes should divide a surface into subsets ('clusters'), but the cuts between the holes can have infinitely many positions. Thus we have to concentrate on the backbone which is known to have, for the bond problem, the same correlation length exponent ν_B as the problem defined through the probability that two sites belong to the same cluster (Shlifer *et al* 1979). The surface backbone can be defined as the union of singly connected self-avoiding surfaces which occur in the system of randomly

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occupied plaquettes (and which should not be confused with 'self-avoiding random surfaces' (Maritan and Stella 1984) and span the system from border to border.

We define the dual problem to surface percolation as follows (Aizenmann *et al* 1983): each plaquette has a bond in its centre perpendicular to it, which is occupied if the plaquette is empty, and vice versa. Therefore this is a bond percolation problem with the occupation probability $p^B = 1 - p^S$. Clearly, if the 'last' bond from the infinite bond cluster is cut by the surface, the surface will span the infinite sample without any holes.

Let us now see the physical meaning of the correlation length ξ_S of the surface problem. At a given value of $p^S < p_c^S$ one can find areas of convex border on surfaces without holes inside. If $p^S \to p_c^S$ there will be surfaces with larger and larger areas. The characteristic radius of these areas will be ξ_S. However, this length is just the characteristic distance between the nodes of the bond backbone in the 'blobs, nodes and links picture' (Stanley and Coniglio 1983), since the surfaces can avoid the finite bond clusters, the dangling ends and blobs of the infinite cluster, but at least on distances of the nodes they must get holes. Thus $\xi_S = \xi_B$, the bond percolation correlation length.

The duality also shows up in the finite size scaling. Let us put the system of randomly occupied plaquettes into a box of linear size L where $L \ll \xi_S$. Then we will find a percolating surface spanning the finite box. However, if $L \sim \xi_S$ we no longer find a spanning surface, since holes appear, but the distance between the holes is the same as that between these bonds in the dual problem which penetrate through the surfaces.

The above argument shows already how the exponent ν_S of the surface problem, defined as $\xi_S \propto |p^S - p_c^S|^{-\nu_S}$, can be calculated. Since this problem is dual to the usual bond percolation, the finite size scaling determination of the correlation length exponent ν_B of the bond problem (using the method of Levinshtein *et al* (1975)) is the same as the derivation of ν_S. In this method, the statistics over the occurrence of the first spanning cluster (or backbone) is taken in a box of size L^3, the width of the distribution scaling as $L^{-1/\nu}$. However, due to duality, the statistics over the first spanning surfaces will be the same as for the bonds and therefore

$$\nu_S = \nu_B. \tag{1}$$

Generally, in d dimensions, one has $d-1$ thresholds, which are defined as the critical points of the occurrence of m-dimensional hypersurfaces ($1 < m < d$, $m = 1$ corresponds to bond percolation, $m = 2$ to surface percolation). The thresholds of the m and $(d-m)$-dimensional problems are again related by duality (Aizenman *et al* 1983). A similar argument to the above would therefore suggest that $\nu_{d-1} = \nu_B$ for general d, where ν_{d-1} is the correlation length exponent of the percolation problem for $(d-1)$-dimensional surfaces in d dimensions, $m = d-1$.

However, this result is valid only up to $d = 6$, the upper critical dimension of bond percolation. From bond percolation we learned that at the upper critical dimension the backbone becomes essentially a random walk, the multiply connected parts no longer being important, and at the threshold there is an infinite number of infinite clusters separated by a length diverging in a different way from the correlation length (Coniglio 1985). Generalising this result, it is expected that above the upper critical dimensions of the m-dimensional percolation, the backbone will be a random object with the topological dimension m. According to the transparency argument (Stanley

1985) the dimension above which two random objects of fractal dimension d_f can avoid each other is $2d_f$. This leads to a lower bound for the upper critical dimension of the m problem: $d_u^{(m)} > 2d_f^{(m)}$, with $d_f^{(m)}$ being the fractal dimension of the random m-hyperface. ($d_f^{(1)} = 2$ and $d_f^{(2)} = 4$ (Parisi 1979). It is only a lower bond, because in general the transparency argument should be applied to the infinite cluster and not to the backbone.) However, this lower bound shows already that the upper critical dimension of the $d-1$ problem in d dimensions should go to infinity when $d \to \infty$. Moreover, since $d_f^{(m)} > m$, $d_f^{(m)} \to \infty$, if $m \to \infty$ and ν_{d-1} cannot be expected to be $\frac{1}{2}$, the mean-field value of the dual ($m = 1$) problem for all $d > 6$.

This virtual contradiction finds its solution in the breakdown of hyperscaling and finite-size scaling (Binder *et al* 1985) above the upper critical dimension. Binder *et al* (1985) showed that there is a second characteristic length l called the thermodynamic length, with an exponent $d_u\nu/d$, which appears beside the correlation length ξ; it is this thermodynamic length which governs finite-size scaling. In the finite-size scaling for percolation this effect has a simple geometrical explanation. Above the upper critical dimension there are infinitely many infinite clusters at the threshold (Coniglio 1985) and the thermodynamic length l is the characteristic distance between these infinite incipient clusters. In a finite-size scaling study of bond percolation (Levinshtein *et al* 1975) the following reasoning can be made. Let us start from above the critical point in a box of size L. Percolation through the box will be impossible if gaps of size L become characteristic. Below the critical dimension d_u, ξ_B describes the gap size between the nodes in the infinite cluster, but above d_u the smaller characteristic length, i.e. l, the distance between the infinite clusters, is to be taken, since we are not interested in holes in one infinite cluster but in the space without any of them.

For the $(d-1)$-dimensional hypersurface percolation the length l is the real correlation length, because, from the point of view of the hypersurface, it does not matter whether the holes were formed by a single or by many infinite bond clusters, obviously the shorter length, i.e. l, is to be taken. Therefore we can conclude that

$$\nu_{d-1} = \nu_B \qquad \text{for } d \leqslant 6 \tag{2a}$$

$$\nu_{d-1} = 6\nu_B/d = 3/d \qquad \text{for } d > 6. \tag{2b}$$

Let us finish with some speculation about the upper critical dimension of the hypersurface problem, especially for the most interesting $m = 2$ case. It is known that the upper critical dimension for the case of self-avoiding surfaces (SAS) is eight (Maritan and Stella 1984). (This result can be obtained by the transparency argument using the fractal dimension of the corresponding random surfaces $d_f^{(2)} = 4$ (Parisi 1979).) Let us suppose that the mean-field correlation length exponent of surface percolation is equal to the reciprocal value of the fractal dimension of the mean-field backbone, i.e. again a random surface. Since the backbone consists essentially of cutting bonds in the mean-field theory and since the number of cutting bonds has an exponent $1/\nu$, this is plausible. In addition, we assume that the relation $d_u^{(m)}/d_f^{(m)} = 3$ is valid, as it is for $m = 1$. If our assumptions are correct we obtain $d_u^{(2)} = 12$. As a possible extension of this result we suggest that generally $d_f^{(m)} = 2m$, as is the case for $m = 1, 2$, and that $d_u^{(m)} = 6m$.

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